The Five Color Theorem Worksheet

In this worksheet we will practice proof by induction by looking at the map coloring problem and proving that all maps can be colored in five colors, which is commonly known as the five color theorem. Some familiarity with graph theory, such as Euler's formula for planar graphs, is assumed.

Definition 1 (Graphs)**.** An *undirected graph*, $G = (V, E)$, is a (discrete) set, V, whose elements are called vertices, and a subset, $E \subseteq \{e \in 2^V \mid |e| = 2\}$, of all sets of vertices with cardinally 2, whose elements are called edges[1]. Moreover, A graph is called *planar* if it can be drawn on the plane in such a way that no edges cross each other.

Definition 2 (Path)**.** A *path* (of length k) in a graph, $G = (V, E)$, is a finite sequence of vertices, $P = (v_1, v_2, \ldots, v_{k+1}) \in V^{k+1}$, such that for every $i \in [k] = \{1, 2, \ldots, k\}$ we have that $\{v_i, v_{i+1}\} \in E$. We call a path a $u - v$ path if $v_1 = u$ and $v_{k+1} = v$.

Problems

Problem 1 (Map Coloring)**.** You are given a map of countries/state and you are asked to color each country/state such that no two countries/states that share a border have the same color (two countries only sharing a corner can be colored the same color, e.g., Colorado and Arizona). We will seek to figure out the minimum number of colors needed for an arbitrary map.

1. Many computer science problems are easier to think of as graph problems. For example, graph coloring, is the problem that asks for an assignment of colors to the vertices such that, for each edge, the vertices have different colors. Explain how to turn the problem of map coloring into a graph coloring problem. Moreover, argue that the constructed graph is planar.

   [Hint: Define V and E explicitly.]

2. Let $f : V \to [k]$ be an assignment of colors to a graph where there are k colors labeled by $[k] := \{1, 2, \ldots, k\}$. Using quantifiers, give the condition for a planar graph being properly colored, i.e., no two adjacent vertices have the same color.

3. We seek to find the minimum numbers of colors needed to color an arbitrary map or its corresponding graph. Give a lower bound on the minimum number of colors needed.

   [Hint: Give an instance of a map/planar graph that requires k colors. We say that k is a lower bound on the number of colors needed to color arbitrary planar graphs.]

4. Show that every map/planar graph can always be colored using 6 colors. We say that 6 is an upper bound on the number of colors needed to color arbitrary planar graphs.

   (a) Argue that every planar graph has at least one vertex with degree at most 5.
   [Hint: Use the handshake lemma, the fact that $2|E| \geq 3|F|$, and Euler's formula for planar graphs, i.e., $|V| - |E| + |F| = 2$, to bound the average degree of the graph.]

   (b) Use induction to prove that every map/planar graph can always be colored using 6 colors.

[1] Often, E is taken to be a symmetric relation over V, i.e., $E \subseteq V \times V$. We use this, somewhat less standard, definition to avoid confusion when talking about the cardinality of the set E and the number of edges in a graph.

5. Show that every map/planar graph can always be colored using 5 colors.

   (a) Let $G = (V, E)$ be a planar graph with a 5-coloring, $f : V \to [5]$. Let $u, v \in V$ be two distinct vertices that are assigned different colors. Give an equivalent condition for when there can not exists a coloring $f' : V \to [5]$ such that $f'(u) = f'(v)$ and $\forall w \in V : f(w) \notin \{f(u), f(v)\} \to f'(w) = f(w)$ that depends on the existence of a particular type of $u - v$ path.

   [Hint: Use f to influence additional properties you wish to place on the $u - v$ path to get an equivalent condition.]

   (b) Strengthen your proof in Part 4b to prove that every map/planar graph can always be colored using 5 colors.

   [Hint: In the inductive step, set up a contradiction to argue that you can always recolor the graph assumed to be 5-colorable by the inductive hypothesis in such a way so that the same inductive step is in Part 4b can be used but for 5 colors. In particular, use Part 5a to argue that if no recoloring exists then the graph isn't planar.]

Remark. To understand what lower and upper bounds mean in the context of this question, I'll give an analogy. Imagine you are a painter, and I'm about to give you a map or a graph that I want you to paint. You know $|V|$ (and maybe also $|E|$), but you don't know what the specific graph is. How many colors should you prepare on your palette?

For a lower bound, we showed that there exists a map/graph that requires 4 colors so you should have at least 4 colors prepared. For an upper bound, we showed that every graph can be colored in 6 (or 5 if you did Part 4b) colors, so you don't need to prepare more colors than that. In fact, there is a "4 color theorem," which states that you only ever need to prepare 4 colors.