Monogamy of Entanglement Bounds and Improved Approximation Algorithms for Qudit Hamiltonians

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Maximal Entanglement Problem

Consider a 2-local Hamiltonian defined over an interaction graph $G = (V, E, w)$ whose local terms are projectors onto a maximally entangled two qudit state, $h_{ab} = |\psi_{ab}\rangle\langle\psi_{ab}|^{ab} \otimes I^{V\setminus\{a,b\}}$ $(ab \in E)$.
We seek to maximize the total energy of $H = \sum_{ab \in E} w_{ab} h_{ab}$:

$$\lambda_{max}(H) = \max_{\rho \in \mathcal{D}((\mathbb{C}^d)^{\otimes n})} \operatorname{tr}(\rho H)$$

Monogamy of Entanglement Bounds

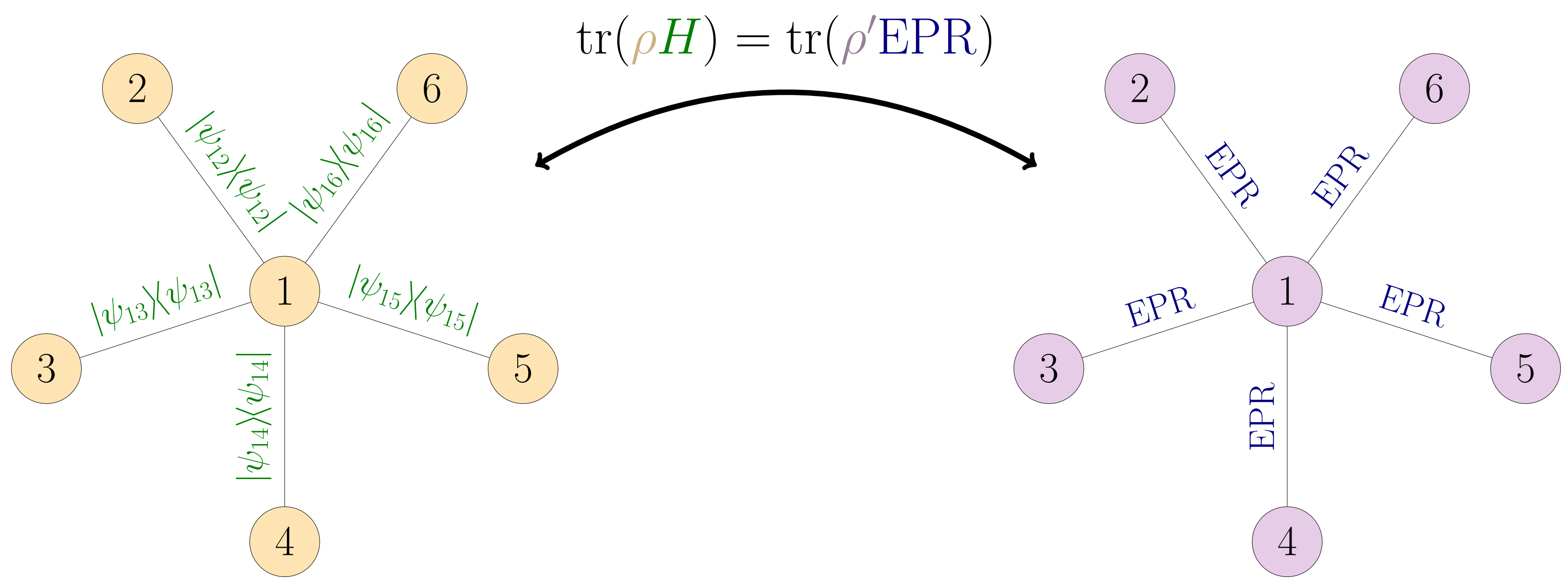
Considering only the edges adjacent to one vertex gives the star graph.

Star Bound

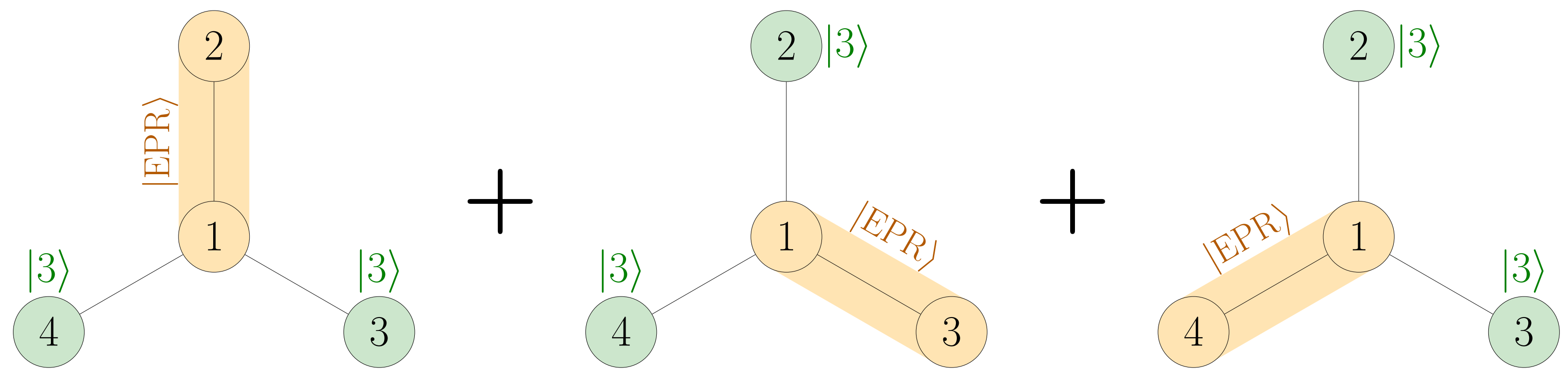
For any state ρ, vertex a, and $S \subseteq N(a)$, we have that

$$\operatorname{tr}\left(\rho \sum_{b \in S} h_{ab}\right) \leq \frac{|S|}{d} + \frac{d-1}{d}$$

(1) The Maximal Entanglement problem reduces to the EPR problem on trees.



(2) There exists a state that matches this energy on the star graph.



While important, the star bound ignores the interactions in low-girth subgraphs. We next consider the interactions in a triangle subgraph.

Triangle Bound

For any state ρ and triangle $S = \{a, b, c\} \subseteq V$, we have that

$$\operatorname{tr}\left(\rho(h_{ab} + h_{bc} + h_{ac})\right) \leq \frac{3}{d} + \frac{d-1}{d}$$

(3) In both bounds, ρ can be a degree-6 pseudo-density matrix.
In particular, the level-3 Lasserre SDP will satisfy these bounds.

Matching-Based Algorithm

The star and triangle bounds certify the energy above $\frac{|S|}{d}$ to be at most $\frac{d-1}{d}$ on both the star and triangle graphs. This gives a (scaled) fractional matching for this additional energy. The results of [Edm65, LP24] imply our main theorem.

Main Theorem

❶ For any state ρ, we have that

$$\operatorname{tr}(\rho H) \leq \frac{W}{d} + \frac{5(d-1)}{4d} M$$

where $W = \sum_{ab \in E} w_{ab}$ and $M = \frac{1}{W} \mathrm{OPT}_{\mathrm{MATCH}}(G)$.

❷ There exists an efficient algorithm that outputs a density matrix ρ such that

$$\operatorname{tr}(\rho H) \geq \left(\frac{1}{d} + \Theta(M)\right) \operatorname{tr}(\rho_* H)$$

where ρ_* is a ground state and $\Theta(M)$ is taken as $M \to 0$.

Algorithm

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Input: $G = (V, E, w)$ and $|\psi_{ab}\rangle$ $(ab \in E)$
  1: Find maximal weight matching $m \subseteq E$
  2: Output: $\rho = \bigotimes_{ab \in m} |\psi_{ab}\rangle\langle\psi_{ab}|^{ab} \otimes \bigotimes_{c \notin m} \frac{1}{d} I^c$
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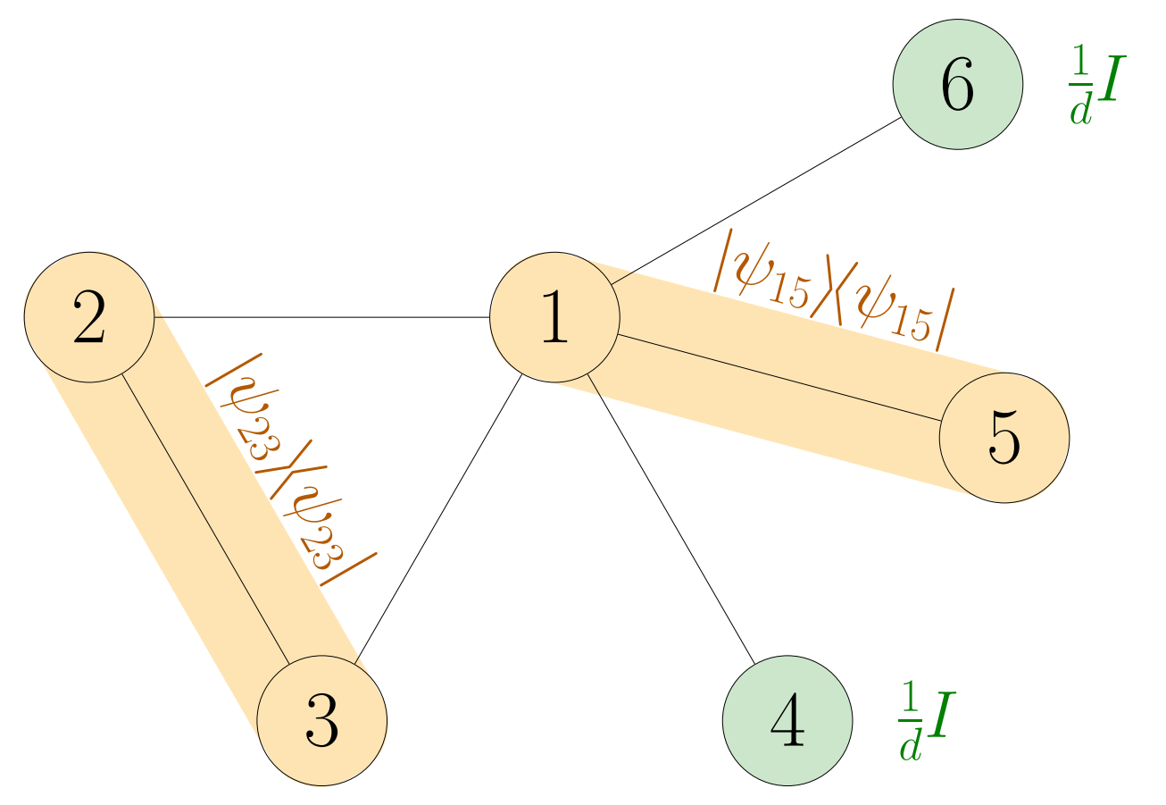

(4) Random assignment (i.e., the maximally mixed state, $\frac{1}{d^n}I$) has an approximation guarantee of $\frac{1}{d} - \Theta(M)$. Therefore, the above matching based algorithm beats random assignment, even when parameterized by M, in all but in the limit as $M \to 0$.

(5) When over qudits $(d = 2)$, we give an algorithm achieving an approximation guarantee of 0.595, beating the 1/2 approximation ratio of [PT22].

References

[Edm65] Jack Edmonds. Maximum matching and a polyhedron with 0,1-vertices. 1965.

[LP24] Eunou Lee and Ojas Parekh. An Improved Quantum Max Cut Approximation via Maximum Matching. 2024.

[PT22] Ojas Parekh and Kevin Thompson. An Optimal Product-State Approximation for 2-Local Quantum Hamiltonians with Positive Terms. 2022.

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